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Asian Resonance

A Note on Prolate Spheroidal Wavelets

Abstract

Prolate Spheroidal wave functions (PSWFs) have been shown to be the best tool for analyzing some problems raised in signal processing and telecommunication. The wavelets derived from PSWFs are most highly localized simultaneously in both the time and frequency domain. Prolate spheroidal wavelets are well behaved with respect to differentiation, translation and convolution. They have also some interesting convergence properties in several function spaces. In this work, wavelet transforms and related properties have been discussed for these new wavelets.

Keywords: Prolate spheroidal wave functions, Paley Wiener space, Vanishing moments, Wavelet transform, Band limited function.

Matth Subject Classification: 42A10, 42A15, 42C40.

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Introduction

The prolate spheroidal wave functions, $\{\phi_{n,\sigma,\tau}(t)\}$, constitute an orthonormal basis of the space of $\sigma-$ band limited functions on the real line. They are concentrated on the interval $[-\tau,\tau]$ and, of course, depend on the two parameters σ and τ. Landau[1],Walter and Shen [7] have characterized them as the Eigen functions of an integral operator:

$$\frac{\sigma}{\pi}\int_{-\tau}^{\tau}\phi_{n,\sigma,\tau}(x)S\left(\frac{\sigma}{\pi}(t-x)\right)dx=\lambda_{n,\sigma,\tau}\phi_{n,\sigma,\tau}(t) \qquad (1.1)$$

The Sinc function $S(t)=\frac{Sin\pi t}{\pi t}$ which appears in this formula is closely related to the PSWFs $\phi_{n,\sigma,\tau}(t)$. In addition to the equation (1.1), the $\phi_{n,\sigma,\tau}(t)$ satisfy an integral equation over $(-\infty,\infty)$ as well :

$$\int_{-\infty}^{\infty}\phi_{n,\sigma,\tau}(x)S_{\sigma}(t-x)dx=\phi_{n,\sigma,\tau}(t) \qquad (1.2)$$

where $S_{\sigma}(t)=\frac{\sigma}{\pi}S\left(\frac{\sigma t}{\pi}\right)$

This leads to a dual orthogonality

$$\int_{-\tau}^{\tau}\phi_{n,\sigma,\tau}(x)\phi_{m,\sigma,\tau}(x)dx=\lambda_{n,\sigma,\tau}\delta_{n,m} \qquad (1.3)$$

and

$$\int_{-\infty}^{\infty}\phi_{n,\sigma,\tau}(x)\phi_{m,\sigma,\tau}(x)dx=\delta_{n,m} \qquad (1.4)$$

Thus, $\{\phi_{n,\sigma,\tau}\}$ constitute an orthogonal basis of $L^2(-\tau,\tau)$, as well as an orthonormal basis of the subspace B_σ of $L^2(-\infty,\infty)$, the Paley Wiener space of all σ- band limited functions.

The PSWFs are closely related to the Fourier transform. Indeed, the Fourier transform of $\phi_{n,\sigma,\tau}$ is given by

Where $K_{\sigma}(\omega)$ is the characteristic function of

[-σ,σ].
Therefore the inverse Fourier transform gives us still another formula:

The above formulae and several other

$$\hat{\phi}_{n,\sigma,\tau}(\omega) = (-1)^n \sqrt{\frac{2\pi\tau}{\sigma\lambda_{n,\sigma,\tau}}}\, \phi_{n,\sigma,\tau}(\frac{\tau\omega}{\sigma}) K_\sigma(\omega)$$

related details may be found in works of several workers (see Landau [1], Landau and Pollak [2&3], Landau and Widom [4] , Papoulis [5], Slepian [6] , Walter and Shen [7] and Walter [8]).

Prolate Spheroidal Wavelets

In order to construct these PSWF wavelets , we begin

$$\hat{\phi}_{n,\sigma,\tau}(t) = (-1)^n \sqrt{\frac{2\pi\tau}{\sigma\lambda_{n,\sigma,\tau}}}\, \frac{1}{2\sigma} \int_{-\sigma}^{\sigma} \phi_{n,\sigma,\tau}(\frac{\tau\omega}{\sigma}) e^{i\omega t} d\omega$$

with a scaling function ϕ, whose integer translates are a Reisz basis of a space V_o . This space is usually taken to be a subspace of $L^2(R)$. We shall take ϕ(x) = $\phi_{o,\pi,\tau}(x)$ where τ is any positive number , with this choice the space V_o will turn out to be the Paley-Wiener space B_π of π- band limited functions no matter what the choice of τ.

There are several ways of constructing bases of the other subspaces $V_m = B_{2^m\pi}$ from those of V_0. One uses the standard wavelet approach in which dilations of $\phi_{o,\pi,\tau}$, that is, $\phi_{o,\pi,\tau}(2^m t)$ are used to get the basis $\phi_{o,\pi,\tau}(2^m t - n)$ of V_m. In this case we get

$$\phi(2^m t) = \phi_{o,\pi,\tau}(2^m t) = 2^{m/2} \phi_{o,2^m\pi,2^{-m}\tau}(t)$$

therefore he concentration interval becomes progressively smaller as m increases. In order to avoid this,we have to find a way to make sure the concentration interval remains constant. We may do this by taking $\phi_{o,2^m\pi,\tau}(t-2^{-m}n)$ as a Riesz basis of V_m.

Thus the PS mother wavelets is given by

$$\psi(t) = Cos(\frac{3\pi}{2}t)\phi_{o,\frac{\pi}{2},\tau}(t)$$

Which is orthogonal to all integer translates of $\phi_{o,\pi,\tau}(t)$.

The PS father wavelet is denoted by

ϕ(t) = $\phi_{o,\pi,\tau}(t)$

Thus we define PS father wavelet at scale m by

$\phi_m(t) = \phi_{o,2^m\pi,\tau}(t)$
.... (2.1)

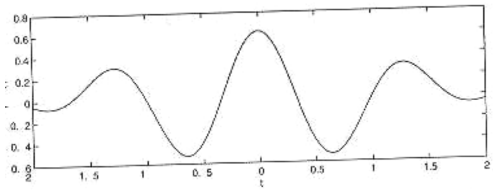
and the PS mother wavelet at scale m is given by

$$\psi_m(t) := Cos(\frac{3\pi}{2} 2^{m-1} t)\phi_{o,2^{m-1}\pi,\tau}(t) \qquad (2.2)$$

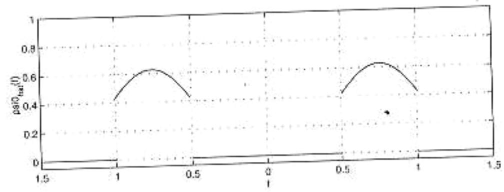
The Translates of mother wavelets form a Riesz basis of the orthogonal complement of V_o in V_1 which is be denoted by W_0 with its dilations denoted by W_m.

We state the following (Theorem 1.in [7])

Let $\{\phi_m\}$ and $\{\psi_m\}$ be given by (2.1) and (2.2), respectively. Let $V_m = B_{2^m\pi}$ and W_m be the orthogonal complement of V_m in V_{m+1}, then $\{\phi_{o,2^m\pi,\tau}(t-2^{-m}n)\}_{n\in z}$ is a Riesz basis of V_m and $\{\psi_m(t-2^{-m}n)\}_{n\in z}$ is a Riesz basis of W_m.



Wavelet $\psi_0(t)$ as defined in (2.2)



$\hat{\psi}_0(\omega)$

Wavelet Transform Properties

`For data compression, denoising and similar applications, the design of ψ(t) must be optimized to produce the maximum number of zero wavelet coefficients. The properties of the wavelet that mostly affect the number of non- zero coefficients are the following :

(i) Number of vanishing moments of the wavelet ψ(t)
(ii) Regularity of the function ψ(t)

The Wavelet ψ(t) has p vanishing moments if

$$\int_{-\infty}^{\infty} t^k \psi(t) dt = 0, \; for \qquad 0 \le k < p$$

It can be shown that if Fourier transform of the wavelet function ψ(t) is p times continuously differentiable at ω = 0 , the following holds :

(a) The wavelet function has p vanishing moments .
(b) $\hat{\psi}(\omega)$ and its first (p-1) derivatives are zero at ω =0

Fourier transform of ψ(t) is given by :

$$\hat{\psi}(w) = \frac{1}{2}[\hat{\phi}_{o,\frac{\pi}{2},\tau}(\omega - \frac{3\pi}{2}) + \hat{\phi}_{o,\frac{\pi}{2},\tau}(\omega + \frac{3\pi}{2})]$$

Moreover the Fourier Transform $\hat{\phi}_{o,\frac{\pi}{2},\tau}(\omega)$ is given by

$$\hat{\phi}_{o,\frac{\pi}{2},\tau}(\omega) = \sqrt{\frac{4\tau}{\lambda_{o,\frac{\pi}{2},\tau}}}\phi_{o,\frac{\pi}{2},\tau}(\frac{2\tau\omega}{\pi})K_{\pi/2}(\omega)$$

This shows that $\hat{\psi}(\omega)$ has a compact support [-2π,-π] U [π,2π] and $\hat{\psi}(\omega)$=0 in the vicinity of ω=0. Hence all derivatives of $\hat{\psi}(\omega)$ are zero at ω=0 and this proves that ψ(t) has infinite number of vanishing moment.

Now let us consider regularity of the function ψ(t).

Since $\psi(t) = Cos(\frac{3\pi}{2}t)\phi_{o,\frac{\pi}{2},\tau}(t)$, by using Leibnitz formula, we may write for k-th derivative :

$$\psi^k(t) = \sum_{r=0}^{k} \binom{k}{r} D^{k-r}\phi_{o,\frac{\pi}{2},\tau}(t) D^r Cos(\frac{3\pi}{2}t)$$

$$= \sum_{r=0}^{k} \binom{k}{r} D^{k-r}\phi_{o,\frac{\pi}{2},\tau}(t)(\frac{3\pi}{2}t)^r Cos(\frac{3\pi}{2}t + \frac{r\pi}{2})$$

Therefore

$$\left|\psi^k(t)\right| \le \sum_{r=0}^{k} \binom{k}{r} \left|D^{k-r}\phi_{o,\frac{\pi}{2},\tau}(t)\right| (\frac{3\pi}{2}t)^r$$

Since $\phi^{(n)}(t)$ is bounded for n=01,2,- - - as proved in [1], Hence $\psi^{(k)}(t)$ exists for all k=01,2,- - - . It is also easy to check that $\psi^{(k)}(t)$ belongs to W_o . Thus we have proved the following proposition:

Proposition 1: The PS mother wavelets ψ(t) has an infinite number of vanishing moments and has derivatives of all orders belonging to the same space W_o.

Integration

In this sub section, we shall consider the integration of the PS wavelet ψ(t). As we have seen above that $\widehat{\Psi}(0) = \int_{-\infty}^{\infty}\Psi(t)dt = 0$. Furthermore, we have the following result about the integration of PS wavelet ψ(t):

Proposition 2 The integral of PS wavelet ψ(t) belongs to the space W_0 which is orthogonal complement of V_0 in V_1.

Proof : Let $g(t) = \int_{-\infty}^{t}\psi(t)dt,$

$$for, \qquad \omega \ne o$$

$$\hat{g}(w) = \int_{-\infty}^{\infty} e^{-i\omega t}\left[\int_{-\infty}^{t}\Psi(u)du\right]dt$$

Integrating by parts we have

$$\hat{g}(w) = \left(\frac{e^{-i\omega t}}{-i\omega}\int_{-\infty}^{t}\Psi(u)du\right)\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\Psi(t)\left(\frac{e^{-i\omega t}}{-i\omega}\right)dt$$

$$= \frac{1}{i\omega}\int_{-\infty}^{\infty}\Psi(t)e^{-i\omega t}dt = \frac{1}{i\omega}\hat{\psi}(\omega) , \qquad \omega \ne o$$

Thus we see that the Fourier transform of the integral $\int_{-\infty}^{t}\psi(t)dt$ has the same support $[-2\pi,-\pi] \cup [\pi,2\pi]$ and hence the integral function g(t) belongs to W_o.

If $\omega = o$, then

$$\hat{g}(0) = \int_{-\infty}^{\infty}\int_{-\infty}^{t}\Psi(u)dtdu$$

Since

$$o = \int_{-\infty}^{\infty}\Psi(t)dt = \int_{-\infty}^{t}\Psi(t)dt + \int_{t}^{\infty}\Psi(t)dt$$

we have $\hat{g}(0) = -\int_{-\infty}^{\infty}\int_{t}^{\infty}\Psi(u)dtdu$

$$= -\int_{-\infty}^{0}\int_{t}^{\infty}\Psi(u)dtdu - \int_{0}^{\infty}\int_{t}^{\infty}\Psi(u)dtdu$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{t}\Psi(u)dtdu - \int_{0}^{\infty}\int_{t}^{\infty}\Psi(u)dtdu$$

$$= \int_{-\infty}^{0}\int_{u}^{0}\Psi(u)dudt - \int_{0}^{\infty}\int_{0}^{u}\Psi(u)dudt$$

$$= -\int_{-\infty}^{0}u\Psi(u)du - \int_{0}^{\infty}u\Psi(u)du$$

$$= -\int_{-\infty}^{\infty}u\Psi(u)du = 0,$$

since $\psi(t)$ has infinite number of vanishing moments. Thus, we see that the Fourier transform of the integral of $\psi(t)$ is zero at $\omega = o$.

Wavelet Transform Coefficients:

As we know that $\left\{\psi_m(t-2^{-m}n) : n \in Z\right\}$ is a Riesz basis of W_m, therefore, the wavelet transform coefficient of f∈ $L^2(R)$ is given by

$$\left\langle f(t), \psi_m(t-2^{-m}n) : n \in Z\right\rangle$$

By Plancharel theorem, we have,

$$\left\langle f(t), \psi_m(t-2^{-m}n)\right\rangle = \frac{1}{2\pi}\left\langle \hat{f}(\omega), \hat{\psi}_m(\omega)e^{-i(2^{-m}n)\omega}\right\rangle$$

Using the definitions (2.1) and (2.2) , we have,

$$\hat{\psi}_m(\omega) = \frac{1}{2}\left[\hat{\phi}_{m-1}(\omega - 3\pi 2^{m-1}) + \hat{\phi}_{m-1}(\omega + 3\pi 2^{m-1})\right]$$

This shows that support of $\hat{\psi}_m$ is $\left[-2^{m+1}\pi, -2^m\pi\right] \cup \left[2^m\pi, 2^{m+1}\pi\right]$ which is same as support of $\hat{\psi}\ (2^{-m}\omega)$. Now let us assume that

$g_m(\omega) = \hat{f}(\omega)\overline{\hat{\psi}\ (2^{-m}\omega)}$ and

$$F_m(\omega) = \sum_{K\in Z} g_m(\omega + 2^{m+1}k\pi)$$

$$= \sum_{K\in Z} \hat{f}(\omega + 2^{m+1}k\pi)\overline{\hat{\psi}\ (2^{-m}\omega + 2k\pi)}$$

Here $F_m(\omega)$ is $(2^{m+1}\pi)$ - periodic and so is the function $e^{-i(2^{-m}n)\omega}$ In fact, the system $\left\{E_n^m = \frac{2^{-m}}{2\pi} e^{-i(2^m n)\omega} : n \in Z\right\}$ is an orthonormal basis for L^2 ([0, $2^{m+1}\pi$]). Thus we obtain

$$\left\langle f(t), \psi_m(t - 2^{-m}n)\right\rangle$$

$$= 2^m \int_R \hat{f}(\omega)\overline{\hat{\psi}(2^{-m}\omega)}\frac{2^{-m}}{2\pi}e^{i(2^{-m}n)\omega}\, d\omega$$

$$= 2^m \int_0^{2^{m+1}\pi} \left(\sum_{K\in Z} g_m(\omega + 2^{m+1}k\pi)\right)\frac{2^{-m}}{2\pi}e^{i(2^m n)\omega}\, d\omega$$

$$= 2^m\left\langle F_m, E_n^m\right\rangle$$

Where the last inner product is the one associated with $L^2([0,2^{m+1}\pi])$. This implies that the wavelet coefficients are zero only when $F_m = 0$.

Now we can prove the following proposition:

Proposition 3 : Suppose that $f \in L^2(R)$ and $\hat{f}$ has support contained in I=(a,b), where $b-a \le 2\pi$ and $I \cap [-\pi,\pi] = \phi$, then it has non zero wavelet coefficients only for some $m \ge 0$ and all the wavelet coefficients of f are zero for $m < 0$.

Proof: It $m \ge 0$, $k \ne 0$ and $\omega \in I$, we have that $(\omega + 2^{m+1}k\pi)$ lies outside the support of $\hat{f}$. In fact, If k > o

$b = a + (b - a) \le a + 2\pi \le a + 2^{m+1}k\pi < \omega + 2^{m+1}k\pi$

so that, in this case , $\omega + 2^{m+1}k\pi$ is to the right of I.

If $k < 0$

$a = b + (a - b) = b - (b - a) \ge b - 2\pi \ge b + 2k\pi \ge b + 2^{m+1}k\pi > \omega + 2^{m+1}k\pi$

This proves that $\omega + 2^{m+1}k\pi$ is to the left of I. Therefore all the terms in F_m are zero except the one corresponding to k = 0

For k = 0 , $\hat{\psi}(2^{-m}\omega + 2k\pi) = \hat{\psi}(2^{-m}\omega) = 0$ only when $\omega \in (-\pi, \pi)$.

If $m < 0$ then $|2^{-m}\omega| = 2^{-m}\,|\omega| > 2^{-m}\pi$ since $\omega \in I$ and $I \cap [-\pi,\pi] = \phi$,

hence $\hat{\psi}(2^{-m}\omega) = 0$ for all m < 0 . If $(\omega + 2^{m+1}k\pi) \in I$,

$|\ 2^{-m}\omega + 2k\pi\ | = 2^{-m}\ |\omega + 2^{m+1}k\pi| = 2^{-m}\,|\omega + 2^{m+1}k\pi| > 2^{-m}\pi \ge 2\pi$

Thus $|\ 2^{-m}\omega + 2k\pi\ | > 2\pi$ for all $m < 0$.

Hence, $\hat{\psi}(2^{-m}\omega + 2k\pi) = 0$ for all $m < 0$.

This completes the proof of the proposition.

The Continuous Wavelet Transform :

Let $a \ne 0$ and b be real numbers. The Continuous PS wavelet transform of $f \in L^2(R)$ can be defined by

$$(W_\psi f)(a,b) = a^{-1/2}\int_{-\infty}^{\infty} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}dt$$

Where ψ is the PS mother wavelets. With

$$(\psi_{a,b})(t) = |a|^{-1/2}\ \psi\left(\frac{t-b}{a}\right)$$

The terms

$(W_\psi f)(a,b) = \langle f, \psi_{a,b}\rangle$ are called the continuous wavelet coefficients of f.

Example 1 :

We shall consider wavelet transform of an harmonic function. Let the signal be given by

$$S(t) = Ae^{i\omega_1 t}$$

Then the wavelet transform is expressed as

$$(W_\psi S)(a,b) = \int_{-\infty}^{\infty} Ae^{-i\omega_1 t}\psi_{a,b}(t)dt = \sqrt{2\pi}A\hat{\psi}_{a,b}(\omega_1)$$

But

$$\hat{\psi}_{a,b}(\omega_1) = \sqrt{a}\hat{\psi}(a\omega_1)e^{-i\omega_1 b}$$

Therefore

$$(W_\psi S)(a,b) = A\sqrt{2\pi a}\hat{\psi}(a\omega_1)e^{-i\omega_1 b}$$

The last expression shows that the CWT of the signal is an analytic function which tends to zero as $b \to \infty$. We can also generalize the above example for the case that the amplitude A is a function of time. In this case, the wavelet transform can be seen as the Fourier transform of the product of two signals, A(t) and $\psi_{a,b}(t)$ which results in convolution of the two Fourier transforms.

Conclusions

We have shown that the PS wavelets are differentiable and integrable functions. Moreover, their differentials and integrals belong to the same space W_0. We have also shown that wavelet coefficients can be determined using their band limited character.

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